NOTES

Electronic Calculators

1. The use of silent electronic calculators is **expected** in Advanced level Mathematics and Further Mathematics (9233 and 9234) and in AO level Mathematics (8174).

2. Graphic calculators will be permitted only in Advanced level Further Mathematics (9234).

3. More detailed regulations concerning the use of electronic calculators will be issued by the Examinations and Assessment Branch of the Ministry of Education.

Lists of Formulae, etc.

Formulae for AO level Mathematics are printed on the question papers.

Candidates entered for A level Mathematics or Further Mathematics will be provided in the examination with a list of formulae.

Mathematical Instruments

Apart from the usual mathematical instruments, candidates may use flexicurves in all the examinations. In addition, candidates taking Advanced level subjects may use stencils for drawing conic sections, provided that the stencils do not bear formulae not included in the list of formulae provided for use in the examination.

Mathematical Notation

Attention is drawn to the list of mathematical notation on pages 9-12.

FURTHER MATHEMATICS (9234)

GCE ADVANCED LEVEL

Syllabus aims and objectives

The aims and objectives for Advanced level Mathematics 9233 apply, with appropriate emphasis.

Scheme of Papers

The examination in Further Mathematics will consist of two three-hour papers, each carrying 50% of the marks, and each marked out of 100.

Paper 1 A paper consisting of about 11 questions of different marks and lengths on Pure Mathematics. Candidates will be expected to answer **all** questions, except for the last question (worth 12 to 14 marks), which will offer two alternatives, only one of which must be answered.

Paper 2 A paper consisting of 4 or 5 questions of different marks and lengths on Mechanics (worth a total of 43 or 44 marks), followed by 4 or 5 questions of different marks and lengths on Statistics (worth a total of 43 or 44 marks), and one final question worth 12 or 14 marks. The final question consists of two alternatives, one on Mechanics and one on Statistics. Candidates will be expected to answer **all** questions, except for the last question, where only one of the alternatives must be answered.

Special Paper

The Further Mathematics Special Paper is withdrawn with effect from year of examination 2002.

Detailed Syllabus

PAPER 1

Knowledge of the syllabus for Pure Mathematics in 9233 is assumed, and candidates may need to apply such knowledge in answering questions.

THEME OR TOPIC		CURRICULUM OBJECTIVES
		Candidates should be able to:
1	Polynomials and rational functions	– recall and use the relations between the roots and coefficients of polynomial equations, for equations of degree 2, 3, 4 only; – use a given simple substitution to obtain an equation whose roots are related in a simple way to those of the original equation;

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		– sketch graphs of simple rational functions, including the determination of oblique asymptotes, in cases where the degree of the numerator and the denominator are at most 2 (detailed plotting of curves will not be required, but sketches will generally be expected to show significant features, such as turning points, asymptotes and intersections with the axes).
2	Polar coordinates	– understand the relations between cartesian and polar coordinates (using the convention $r \geqslant 0$), and convert equations of curves from cartesian to polar form and *vice versa*; – sketch simple polar curves, for $0 \leqslant \theta < 2\pi$ or $-\pi < \theta \leqslant \pi$ or a subset of either of these intervals (detailed plotting of curves will not be required, but sketches will generally be expected to show significant features, such as symmetry, the form of the curve at the pole and least/greatest values of r); – recall the formula $\frac{1}{2}\int_{\alpha}^{\beta} r^2 \mathrm{d}\theta$ for the area of a sector, and use this formula in simple cases.
3	Summation of series	– use the standard results for $\sum r$, $\sum r^2$, $\sum r^3$ to find related sums; – use the method of differences to obtain the sum of a finite series, e.g. by expressing the general term in partial fractions; – recognise, by direct consideration of a sum to n terms, when a series is convergent, and find the sum to infinity in such cases.
4	Further mathematical induction	– use the method of mathematical induction to establish a given result (questions set may involve divisibility tests and inequalities, for example); – recognise situations where conjecture based on a limited trial followed by inductive proof is a useful strategy, and carry this out in simple cases e.g. to find the nth derivative of xe^x.
5	Differentiation and integration	– obtain an expression for $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in cases where the relation between y and x is defined implicitly or parametrically; – derive and use reduction formulae for the evaluation of definite integrals in simple cases; – use integration to find mean values and centroids of two- and three-dimensional figures (where equations are expressed in cartesian coordinates, including the use of a parameter), using strips, discs or shells as appropriate, arc lengths (for curves with equations in cartesian coordinates, including the use of a parameter, or in polar coordinates),

surface areas of revolution about one of the axes (for curves with equations in cartesian coordinates, including the use of a parameter, but not for curves with equations in polar coordinates).

6	Differential equations	– recall the meaning of the terms ‘complementary function’ and ‘particular integral’ in the context of linear differential equations, and recall that the general solution is the sum of the complementary function and a particular integral; – find the complementary function for a second order linear differential equation with constant coefficients; – recall the form of, and find, a particular integral for a second order linear differential equation in the cases where a polynomial or ae^{bx} or $a \cos px + b \sin px$ is a suitable form, and in other simple cases find the appropriate coefficient(s) given a suitable form of particular integral; – use a substitution to reduce a given differential equation to a second order linear equation with constant coefficients; – use initial conditions to find a particular solution to a differential equation, and interpret a solution in terms of a problem modelled by a differential equation.
7	Complex numbers	– understand de Moivre’s theorem, for a positive integral exponent, in terms of the geometrical effect of multiplication of complex numbers; – prove de Moivre’s theorem for a positive integral exponent; – use de Moivre’s theorem for positive integral exponent to express trigonometrical ratios of multiple angles in terms of powers of trigonometrical ratios of the fundamental angle; – use de Moivre’s theorem, for a positive or negative rational exponent in expressing powers of $\sin\theta$ and $\cos\theta$ in terms of multiple angles, in the summation of series, in finding and using the nth roots of unity.
8	Vectors	– use the equation of a plane in any of the forms $ax + by + cz = d$ or $\mathbf{r.n} = p$ or $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, and convert equations of planes from one form to another as necessary in solving problems; – recall that the vector product $\mathbf{a} \times \mathbf{b}$ of two vectors can be expressed either as $\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin\theta\,\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector, or in component form as $(a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$; – use equations of lines and planes, together with scalar and vector products where appropriate, to solve problems concerning distances, angles and intersections, including determining whether a line lies in a plane, is parallel to a plane or intersects a plane, and finding the point of intersection of a line and a plane when it exists,

		finding the perpendicular distance from a point to a plane, finding the angle between a line and a plane, and the angle between two planes, finding an equation for the line of intersection of two planes, calculating the shortest distance between two skew lines, finding an equation for the common perpendicular to two skew lines.
9	Matrices and linear spaces	– recall and use the axioms of a linear (vector) space (restricted to spaces of finite dimension over the field of real numbers only); – understand the idea of linear independence, and determine whether a given set of vectors is dependent or independent; – understand the idea of the subspace spanned by a given set of vectors; – recall that a basis for a space is a linearly independent set of vectors that spans the space, and determine a basis in simple cases; – recall that the dimension of a space is the number of vectors in a basis; – understand the use of matrices to represent linear transformations from $\mathbb{R}^n \to \mathbb{R}^m$; – understand the terms 'column space', 'row space','range space' and 'null space', and determine the dimensions of, and bases for, these spaces in simple cases; – determine the rank of a square matrix, and use (without proof) the relation between the rank, the dimension of the null space and the order of the matrix; – use methods associated with matrices and linear spaces in the context of the solution of a set of linear equations; – evaluate the determinant of a square matrix and find the inverse of a non-singular matrix (2×2 and 3×3 matrices only), and recall that the columns (or rows) of a square matrix are independent if and only if the determinant is non-zero; – understand the terms 'eigenvalue' and 'eigenvector', as applied to square matrices; – find eigenvalues and eigenvectors of 2×2 and 3×3 matrices (restricted to cases where the eigenvalues are real and distinct); – express a matrix in the form $\mathbf{QDQ}^{-1}$, where **D** is a diagonal matrix of eigenvalues and **Q** is a matrix whose columns are eigenvectors, and use this expression, e.g. in calculating powers of matrices.

PAPER 2

Knowledge of the syllabuses for the Particle Mechanics (Sections 1 to 8) and Probability and Statistics (Sections 1 to 7) in Mathematics 9233 is assumed. Candidates may need to apply such knowledge in answering questions; harder questions on those sections may also be set.

THEME OR TOPIC	CURRICULUM OBJECTIVES
	Candidates should be able to:
MECHANICS (Sections 1 to 5)	
1 Momentum and impulse	– recall and use the definition of linear momentum, and show understanding of its vector nature (in one dimension only); – recall Newton's experimental law and the definition of the coefficient of restitution, the property $0 \leqslant e \leqslant 1$, and the meaning of the terms 'perfectly elastic' ($e = 1$) and 'inelastic' ($e = 0$); – use conservation of linear momentum and/or Newton's experimental law to solve problems that may be modelled as the direct impact of two smooth spheres or the direct or oblique impact of a smooth sphere with a fixed surface. – recall and use the definition of the impulse of a constant force, and relate the impulse acting on a particle to the change of momentum of the particle (in one dimension only).
2 Circular motion	– recall and use the radial and transverse components of acceleration for a particle moving in a circle with variable speed; – solve problems which can be modelled by the motion of a particle in a vertical circle without loss of energy (including finding the tension in a string or a normal contact force, locating points at which these are zero, and conditions for complete circular motion).
3 Equilibrium of a rigid body under coplanar forces	– understand and use the result that the effect of gravity on a rigid body is equivalent to a single force acting at the centre of mass of the body, and identify the centre of mass by considerations of symmetry in suitable cases; – calculate the moment of a force about a point in 2 dimensional situations only (understanding of the vector nature of moments is not required); – recall that if a rigid body is in equilibrium under the action of coplanar forces then the vector sum of the forces is zero and the sum of the moments of the forces about any point is zero, and the converse of this; – use Newton's third law in situations involving the contact of rigid bodies in equilibrium;

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		– solve problems involving the equilibrium of rigid bodies under the action of coplanar forces (problems set will not involve complicated trigonometry).
4	Rotation of a rigid body	– understand and use the definition of the moment of inertia of a system of particles about a fixed axis as $\sum mr^2$, and the additive property of moment of inertia for a rigid body composed of several parts (the use of integration to find moments of inertia will not be required); – use the parallel and perpendicular axes theorems (proofs of these theorems will not be required); – recall and use the equation of angular motion $C = I\ddot{\theta}$ for the motion of a rigid body about a fixed axis (simple cases only, where the moment C arises from constant forces such as weights or the tension in a string wrapped around the circumference of a flywheel; knowledge of couples is not included and problems will not involve consideration or calculation of forces acting at the axis of rotation); – recall and use the formula $\frac{1}{2}I\omega^2$ for the kinetic energy of a rigid body rotating about a fixed axis; – use conservation of energy in solving problems concerning mechanical systems where rotation of a rigid body about a fixed axis is involved.
5	Simple harmonic motion	– recall a definition of SHM and understand the concepts of period and amplitude; – use standard SHM formulae in the course of solving problems; – set up the differential equation of motion in problems leading to SHM, recall and use appropriate forms of solution, and identify the period and amplitude of the motion; – recognise situations where an exact equation of motion may be approximated by an SHM equation, carry out necessary approximations (e.g. small angle approximations or binomial approximations) and appreciate the conditions necessary for such approximations to be useful.
STATISTICS (Sections 6 to 9)		
6	Further work on distributions	– use the definition of the distribution function as a probability to deduce the form of a distribution function in simple cases, e.g. to find the distribution function for Y, where $Y = X^3$ and X has a given distribution; – understand conditions under which a geometric distribution or negative exponential distribution may be a suitable probability model; – recall and use the formula for the calculation of geometric or negative exponential probabilities;

		– recall and use the means and variances of a geometric distribution and negative exponential distribution.
7	Inference using normal and *t*-distributions	– formulate hypotheses and apply a hypothesis test concerning the population mean using a small sample drawn from a normal population of unknown variance, using a *t*-test; – calculate a pooled estimate of a population variance from two samples (calculations based on either raw or summarised data may be required); – formulate hypotheses concerning the difference of population means, and apply, as appropriate, a 2-sample *t*-test, a paired sample *t*-test, a test using a normal distribution, (the ability to select the test appropriate to the circumstances of a problem is expected); – determine a confidence interval for a population mean, based on a small sample from a normal population with unknown variance, using a *t*-distribution; – determine a confidence interval for a difference of population means, using a *t*-distribution or a normal distribution, as appropriate.
8	χ^2-tests	– fit a theoretical distribution, as prescribed by a given hypothesis, to given data (questions will not involve lengthy calculations); – use a χ^2-test, with the appropriate number of degrees of freedom, to carry out the corresponding goodness of fit analysis (classes should be combined so that each expected frequency is at least 5); – use a χ^2-test, with the appropriate number of degrees of freedom, for independence in a contingency table (Yates' correction is not required, but classes should be combined so that the expected frequency in each cell is at least 5).
9	Bivariate data	– understand the concept of least squares, regression lines and correlation in the context of a scatter diagram; – calculate, both from simple raw data and from summarised data, the equations of regression lines and the product moment correlation coefficient, and appreciate the distinction between the regression line of y on x and that of x on y; – recall and use the facts that both regression lines pass through the mean centre $(\bar{x}, \bar{y})$ and that the product moment correlation coefficient r and the regression coefficients b_1, b_2 are related by $r^2 = b_1 b_2$; – select and use, in the context of a problem, the appropriate regression line to estimate a value, and understand the uncertainties associated with such estimations;

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- relate, in simple terms, the value of the product moment correlation coefficient to the appearance of the scatter diagram, with particular reference to the interpretation of cases where the value of the product moment correlation coefficient is close to +1, – 1 or 0;
- carry out a hypothesis test based on the product moment correlation coefficient.

MATHEMATICAL NOTATION

The list which follows summarizes the notation used in the Syndicate's Mathematics examinations. Although primarily directed towards Advanced level, the list also applies, where relevant, to examinations at all other levels, i.e. O level, AO level and N level.

Mathematical Notation

1. *Set Notation*

$\in$	is an element of
$\notin$	is not an element of
$\{x_1, x_2, \ldots\}$	the set with elements $x_1, x_2, \ldots$
$\{x: \ldots\}$	the set of all x such that . . .
$n(A)$	the number of elements in set A
$\varnothing$	the empty set
$\mathscr{E}$	universal set
A'	the complement of the set A
$\mathbb{N}$	the set of positive integers, $\{1, 2, 3, \ldots\}$
$\mathbb{Z}$	the set of integers $\{0, \pm1, \pm2, \pm3, \ldots\}$
$\mathbb{Z}^+$	the set of positive integers $\{1, 2, 3, \ldots\}$
$\mathbb{Z}_n$	the set of integers modulo n, $\{0, 1, 2, \ldots, n-1\}$
$\mathbb{Q}$	the set of rational numbers
$\mathbb{Q}^+$	the set of positive rational numbers, $\{x \in \mathbb{Q}: x > 0\}$
$\mathbb{Q}_0^+$	the set of positive rational numbers and zero, $\{x \in \mathbb{Q}: x \geqslant 0\}$
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^+$	the set of positive real numbers $\{x \in \mathbb{R}: x > 0\}$
$\mathbb{R}_0^+$	the set of positive real numbers and zero $\{x \in \mathbb{R}: x \geqslant 0\}$
$\mathbb{R}^n$	the real n tuples
$\mathbb{C}$	the set of complex numbers
$\subseteq$	is a subset of
$\subset$	is a proper subset of
$\nsubseteq$	is not a subset of
$\not\subset$	is not a proper subset of
$\cup$	union
$\cap$	intersection
$[a, b]$	the closed interval $\{x \in \mathbb{R}: a \leqslant x \leqslant b\}$
$[a, b)$	the interval $\{x \in \mathbb{R}: a \leqslant x < b\}$
$(a, b]$	the interval $\{x \in \mathbb{R}: a < x \leqslant b\}$
(a, b)	the open interval $\{x \in \mathbb{R}: a < x < b\}$
yRx	y is related to x by the relation R
$y \sim x$	y is equivalent to x, in the context of some equivalence relation

2. *Miscellaneous Symbols*

$=$	is equal to
$\neq$	is not equal to
$\equiv$	is identical to or is congruent to
$\approx$	is approximately equal to
$\cong$	is isomorphic to
$\propto$	is proportional to
$<$; $\ll$	is less than; is much less than
$\leqslant$, $\ngtr$	is less than or equal to or is not greater than
$>$; $\gg$	is greater than; is much greater than
$\geqslant$, $\nless$	is greater than or equal to or is not less than
∞	infinity

MATHEMATICAL NOTATION

3. *Operations*

$a + b$	a plus b
$a - b$	a minus b
$a \times b$, ab, $a.b$	a multiplied by b
$a \div b$, $\frac{a}{b}$, a/b	a divided by b
$a : b$	the ratio of a to b
$\sum_{i=1}^{n} a_i$	$a_1 + a_2 + \ldots + a_n$
$\sqrt{a}$	the positive square root of the real number a
$\lvert a \rvert$	the modulus of the real number a
$n!$	n factorial for $n \in \mathbb{N}$ $(0! = 1)$
$\binom{n}{r}$	the binomial coefficient $\frac{n!}{r!\,(n-r)!}$, for $n, r \in \mathbb{N}$, $0 \leqslant r \leqslant n$ $\frac{n(n-1)\ldots(n-r+1)}{r!}$, for $n \in \mathbb{Q}$, $r \in \mathbb{N}$

4. *Functions*

f	function f
f(x)	the value of the function f at x
f : $A \to B$	f is a function under which each element of set A has an image in set B
f : $x \mapsto y$	the function f maps the element x to the element y
f^{-1}	the inverse of the function f
$\mathrm{g} \circ \mathrm{f}$, gf	the composite function of f and g which is defined by $(\mathrm{g} \circ \mathrm{f})(x)$ or $\mathrm{gf}(x) = \mathrm{g}(\mathrm{f}(x))$
$\lim_{x \to a} \mathrm{f}(x)$	the limit of f(x) as x tends to a
$\Delta' x$; δx	an increment of x
$\frac{\mathrm{d}y}{\mathrm{d}x}$	the derivative of y with respect to x
$\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$	the nth derivative of y with respect to x
$\mathrm{f}'(x), \mathrm{f}''(x), \ldots, \mathrm{f}^{(n)}(x)$	the first, second, . . . , nth derivatives of f(x) with respect to x
$\int y\,\mathrm{d}x$	indefinite integral of y with respect to x
$\int_a^b y\,\mathrm{d}x$	the definite integral of y with respect to x for values of x between a and b
$\frac{\partial y}{\partial x}$	the partial derivative of y with respect to x
$\dot{x}, \ddot{x}, \ldots$	the first, second, . . . derivatives of x with respect to time.

5. *Exponential and Logarithmic Functions*

e	base of natural logarithms
e^x, $\exp x$	exponential function of x
$\log_a x$	logarithm to the base a of x
$\ln x$	natural logarithm of x
$\lg x$	logarithm of x to base 10

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6. *Circular and Hyperbolic Functions and Relations*

sin, cos, tan, cosec, sec, cot	the circular functions
$\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, cosec^{-1}, $\sec^{-1}$, $\cot^{-1}$	the inverse circular relations
sinh, cosh, tanh, cosech, sech, coth	the hyperbolic functions
$\sinh^{-1}$, $\cosh^{-1}$, $\tanh^{-1}$, cosech^{-1}, sech^{-1}, $\coth^{-1}$	the inverse hyperbolic relations

7. *Complex Numbers*

i	square root of –1
z	a complex number, $z = x + \mathrm{i}y$ $= r(\cos\theta + \mathrm{i}\sin\theta)$, $r \in \mathbb{R}_0^+$ $= r\mathrm{e}^{\mathrm{i}\theta}$, $r \in \mathbb{R}_0^+$
Re z	the real part of z, $\mathrm{Re}(x + \mathrm{i}y) = x$
Im z	the imaginary part of z, $\mathrm{Im}(x + \mathrm{i}y) = y$
$\lvert z \rvert$	the modulus of z, $\lvert x + \mathrm{i}y \rvert = \sqrt{(x^2 + y^2)}$, $\lvert r(\cos\theta + \mathrm{i}\sin\theta) \rvert = r$
arg z	the argument of z, $\arg(r(\cos\theta + \mathrm{i}\sin\theta)) = \theta$, $-\pi < \theta \leqslant \pi$
z^*	the complex conjugate of z, $(x + \mathrm{i}y)^* = x - \mathrm{i}y$

8. *Matrices*

$\mathbf{M}$	a matrix $\mathbf{M}$
$\mathbf{M}^{-1}$	the inverse of the square matrix $\mathbf{M}$
$\mathbf{M}^{\mathrm{T}}$	the transpose of the matrix $\mathbf{M}$
det $\mathbf{M}$	the determinant of the square matrix $\mathbf{M}$

9. *Vectors*

$\mathbf{a}$	the vector $\mathbf{a}$
$\overrightarrow{AB}$	the vector represented in magnitude and direction by the directed line segment AB
$\hat{\mathbf{a}}$	a unit vector in the direction of the vector $\mathbf{a}$
$\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$	unit vectors in the directions of the cartesian coordinate axes
$\lvert \mathbf{a} \rvert$	the magnitude of $\mathbf{a}$
$\lvert \overrightarrow{AB} \rvert$	the magnitude of $\overrightarrow{AB}$
$\mathbf{a} \cdot \mathbf{b}$	the scalar product of $\mathbf{a}$ and $\mathbf{b}$
$\mathbf{a} \times \mathbf{b}$	the vector product of $\mathbf{a}$ and $\mathbf{b}$

10. *Probability and Statistics*

A, B, C, etc.	events
$A \cup B$	union of events A and B
$A \cap B$	intersection of the events A and B
$\mathrm{P}(A)$	probability of the event A
A'	complement of the event A, the event ‘not A’
$\mathrm{P}(A \mid B)$	probability of the event A given the event B
X, Y, R, etc.	random variables
x, y, r, etc.	values of the random variables X, Y, R, etc.
$x_1, x_2, \ldots$	observations
$f_1, f_2, \ldots$	frequencies with which the observations $x_1, x_2, \ldots$ occur
$\mathrm{p}(x)$	the value of the probability function $\mathrm{P}(X = x)$ of the discrete random variable X

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$p_1, p_2, \ldots$	probabilities of the values $x_1, x_2, \ldots$ of the discrete random variable X.
$\mathrm{f}(x), \mathrm{g}(x), \ldots$	the value of the probability density function of the continuous random variable X
$\mathrm{F}(x), \mathrm{G}(x), \ldots$	the value of the (cumulative) distribution function $\mathrm{P}(X \leqslant x)$ of the random variable X
$\mathrm{E}(X)$	expectation of the random variable X
$\mathrm{E}[\mathrm{g}(X)]$	expectation of $\mathrm{g}(X)$
$\mathrm{Var}(X)$	variance of the random variable X
$\mathrm{G}(t)$	the value of the probability generating function for a random variable which takes integer values
$\mathrm{B}(n, p)$	binomial distribution, parameters n and p
$\mathrm{N}(\mu, \sigma^2)$	normal distribution, mean μ and variance σ^2
μ	population mean
σ^2	population variance
σ	population standard deviation
$\overline{x}$	sample mean
s^2	unbiased estimate of population variance from a sample, $s^2 = \frac{1}{n-1}\sum(x - \overline{x})^2$
ϕ	probability density function of the standardised normal variable with distribution N(0, 1)
Φ	corresponding cumulative distribution function
ρ	linear product-moment correlation coefficient for a population
r	linear product-moment correlation coefficient for a sample
$\mathrm{Cov}(X, Y)$	covariance of X and Y